TWO NOTES ON THE VARIETY GENERATED BY PLANAR MODULAR LATTICES

GÁBOR CZÉDLI AND MIKLÓS MARÓTI

Dedicated to the memory of András P. Huhn

Abstract. Let $\mathbf{Var}(\mathbf{M}_{\text{plan}})$ denote the variety generated by the class $\mathbf{M}_{\text{plan}}$ of planar modular lattices. In 1977, based on his structural investigations, R. Freese proved that $\mathbf{Var}(\mathbf{M}_{\text{plan}})$ has continuumly many subvarieties. The present paper provides a new approach to this result utilizing lattice identities. We also show that each subvariety of $\mathbf{Var}(\mathbf{M}_{\text{plan}})$ is generated by its planar (subdirectly irreducible) members.

1. Introduction

Let $\mathbf{M}_{\text{plan}}$ denote the class of planar modular lattices. Recall that a lattice L is *planar*, if $(L;\leq)$ is a suborder of the direct square of a finite chain. (Hence planar lattices are *finite* by definition.) Let $\mathbf{Var}(\mathbf{M}_{\text{plan}})$ stand for the variety generated by $\mathbf{M}_{\text{plan}}$. This variety has been intensively studied by G. Grätzer and R. W. Quackenbush [9]. In particular, [9] completely describes the subdirectly irreducible members of $\mathbf{Var}(\mathbf{M}_{\text{plan}})$.

In this paper, we explicitely construct a set $\Sigma = \{\lambda_n : 2 < n \in \mathbf{N}\}$ of *lattice identities* with the property that for any $\Delta \subseteq \Sigma$, the variety $\mathbf{Var}(\Delta) \cap \mathbf{Var}(\mathbf{M}_{\text{plan}})$ determines Δ. To show that Σ has this property, we also construct lattices $L_n \in \mathbf{M}_{\text{plan}}$, for $2 < n$, satisfying the following statement.

Theorem 1. *For $2 < k, n \in \mathbf{N}$, the identity λ_n holds in L_k if and only if $k \neq n$.*

This theorem implies the following classical result of R. Freese [4], which is based on his deep *structural investigations* of modular lattices of width four.

Corollary 2 (R. Freese [4])**.** $\mathbf{Var}(\mathbf{M}_{\text{plan}})$ *has continuumly many subvarieties. In fact, the power set of the set* $\mathbf{N}$ *of natural numbers is a suborder of the lattice of subvarieties of* $\mathbf{Var}(\mathbf{M}_{\text{plan}})$.

The following result states an interesting property of $\mathbf{Var}(\mathbf{M}_{\text{plan}})$.

Theorem 3. *Each subvariety of* $\mathbf{Var}(\mathbf{M}_{\text{plan}})$ *is generated by its subdirectly irreducible planar members.*

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Notation. We use the notation of G. Grätzer [7]. The Glossary of Notation of [7] is available as a pdf file at

`http://mirror.ctan.org/info/examples/Math_into_LaTeX-4/notation.pdf`

2. Proof of Theorem 3

The structure of planar modular lattices is well-understood, see G. Grätzer and R. W. Quackenbush [9]. Let M_n denote the modular lattice of length two with exactly n atoms, $n \geq 3$. Let L be a planar modular lattice. Using a result of Jónsson [13], it is shown in [9] that each M_n-sublattice of L is necessarily a *covering sublattice of* L, that is, if $x \prec y$ in M_n, then $x \prec y$ in L. By removing the inner elements from all M_n-sublattices, we obtain a distributive lattice $\mathrm{Frame}\, L$, called the *frame of* L. Notice that if we think of L as a fixed diagram in the plane, then $\mathrm{Frame}\, L$ is uniquely defined; otherwise it is unique only up to isomorphism.

In order to reduce the complexity of some formulas, the join and meet of elements x and y in a lattice will be denoted by $x + y$ and xy, respectively. Every lattice in this paper is assumed to be modular.

We prove four lemmas we shall need to verify Theorem 1. The first lemma is part of the folklore; we prove it for the reader's convenience.

Lemma 4. *Let Θ be a congruence of a lattice L such that L/Θ is finite. Then we can select an element a_B in each $B \in L/\Theta$ such that the mapping $\varphi\colon B \mapsto a_B$ is an order embedding of L/Θ in L.*

Proof. First, select an element c_B in each $B \in L/\Theta$. Then we define the a_B by induction. For the zero Z of L/Θ, let $a_Z = c_Z$. If $B \in L/\Theta$ is distinct from Z and $a_C \in C$ has already been defined for all $C < B$, then we define $a_B = c_B + \sum\{a_C : C < B\}$. This element belongs to B, because Θ is a congruence. Clearly, φ is injective and isotone. Further, if $D \not\leq B$, then $a_D + a_B \in D + B \neq B \ni a_B$ guarantees that $a_D \not\leq a_B$. $\square$

The following lemma is very easy and can also be found in G. Grätzer and R. W. Quackenbush [10].

Lemma 5. $\mathbf{Var}(\mathbf{M}_{\mathrm{plan}})$ *is a locally finite variety.*

Proof. Let $L \in \mathbf{Var}(\mathbf{M}_{\mathrm{plan}})$ be generated by $\{a_1, \dots, a_n\}$. Let $FD(2n)$ denote the free distributive lattice on $2n$ generators; it is a finite lattice. If a_i is an inner element of an M_n-sublattice, then let b_i and c_i be its lower and upper cover, respectively. Otherwise let $b_i = c_i = a_i$. Since the b_i and the c_i belong to $\mathrm{Frame}\, L$, $|[b_1, \dots, b_n, c_1, \dots, c_n]| \leq |FD(2n)|$. Keeping in mind that the inner elements of M_n-sublattices are doubly-irreducible, we conclude that $L = [b_1, \dots, b_n, c_1, \dots, c_n] \cup \{a_1, \dots, a_n\}$, so it has at most $|FD(2n)| + n$ elements. $\square$

Let us call a lattice $L \in \mathbf{Var}(\mathbf{M}_{\mathrm{plan}})$ *locally planar* iff all of its finite sublattices are planar. For example, the direct square of any infinite chain is locally planar but not planar, and it belongs to $\mathbf{Var}(\mathbf{M}_{\mathrm{plan}})$.

Lemma 6. *Every homomorphic image of a locally planar lattice in* $\mathbf{Var}(\mathbf{M}_{\mathrm{plan}})$ *is locally planar.*

Proof. Let $K \in \mathbf{Var}(\mathbf{M}_{\text{plan}})$ be locally planar and let $L = K/\Theta$. Consider an arbitrary finite sublattice L' of L. Then $L' = K'/\Theta\rceil_{K'}$ for an appropriate sublattice K' of K. We know from Lemma 4 that L' is order-isomorphic to a finite subset B of K'. Since $[B]$, which is a finite sublattice of K by Lemma 5, is planar, we conclude that $[B]$, and therefore L', can be embedded in the direct square of a finite chain. Thus, L' is a planar lattice. $\square$

Lemma 7. *The class of all locally planar modular lattices is axiomatizable in* $\mathbf{Var}(\mathbf{M}_{\text{plan}})$.

Proof. For each $n \in \mathbf{N}$, there are only finitely many planar lattices of size n. Hence there is a first order formula Φ_n expressing that whenever $x_1, \dots, x_n$ are pairwise distinct elements, then either they form a planar sublattice or they do not form a sublattice. Now for each $L \in \mathbf{Var}(\mathbf{M}_{\text{plan}})$, the lattice L satisfies $\{\Phi_n : n \in \mathbf{N}\}$ iff L is locally planar. $\square$

Proof of Theorem 3. Let $\mathbf{U}$ be a subvariety of $\mathbf{Var}(\mathbf{M}_{\text{plan}})$. We show that the class of subdirectly irreducible members of $\mathbf{U}$ in $\mathbf{M}_{\text{plan}}$, that is, $\mathbf{Si}(\mathbf{U}) \cap \mathbf{M}_{\text{plan}}$, generates $\mathbf{U}$. Since $\mathbf{U}$, like any variety, is generated by $\mathbf{Si}(\mathbf{U})$, it suffices to show that an arbitrary $K \in \mathbf{Si}(\mathbf{U})$ is in the variety generated by $\mathbf{Si}(\mathbf{U}) \cap \mathbf{M}_{\text{plan}}$. Since K belongs to the variety generated by $\mathbf{M}_{\text{plan}}$, Jónsson's Lemma, see [12] or [6], yields that $K \in \mathbf{HSP}_u\mathbf{M}_{\text{plan}}$.

By Lemma 7 and the Łoś Theorem, $\mathbf{P}_u\mathbf{M}_{\text{plan}}$ consists of locally planar lattices. Hence, by Lemma 6, K is locally planar. This implies that if a lattice identity λ holds in all finite sublattices of K, then λ holds in K as well. In other words, if a variety contains all finite sublattices of K, then it also contains K. Hence it suffices to show that every finite sublattice K' of K belongs to the variety generated by $\mathbf{Si}(\mathbf{U}) \cap \mathbf{M}_{\text{plan}}$. But this is clear by Lemma 6: the subdirectly irreducible factors of K_i are in $\mathbf{Si}(\mathbf{U}) \cap \mathbf{M}_{\text{plan}}$, for they are homomorphic images of K_i. $\square$

3. The construction of L_n and λ_n

If $x, y, z \in L \in \mathbf{M}_{\text{plan}}$ are the three *distinct* atoms of an M_3-sublattice of L, then the set $\{x, y, z\}$ will be called a *diamond* of L. Since we assume a fixed planar diagram of L, we call a *vector* $\vec{u} = (x, y, z)$ a *diamond* if $\{x, y, z\}$ is a diamond with leftmost (=western) element x, middle element y, and rightmost (=eastern) element z. If $x = y = z$, then we say that $\vec{u}$ is a singleton.

For $\vec{u} = (x, y, z) \in L^3$, for a modular lattice L, let us define

$$\vec{\mathbf{d}}(\vec{u}) = \big((x + yz)(y + z)\,, (y + xz)(x + z)\,, (z + xy)(x + y)\big). \tag{1}$$

Notice that the first component of $\vec{\mathbf{d}}(\vec{u})$ equals $\big(x+(xy+xz+yz)\big)(x+y)(x+z)(y+z)$, and similar terms are use for the other two components. The next lemma follows from R. Dedekind's description of the free modular lattice on three generators [3], see the remark prior to Corollary II.1.3 in G. Grätzer [6].

Lemma 8. *Let* $\vec{u} = (x, y, z) \in L^3$. *Then* $\vec{\mathbf{d}}(\vec{u})$ *is a diamond or a singleton. If* $\vec{u}$ *is a diamond, then* $\vec{\mathbf{d}}(\vec{u}) = \vec{u}$. *If* $\vec{u}$ *is not an antichain, then* $\vec{\mathbf{d}}(\vec{u})$ *is a singleton.*

We would like to point out that A. P. Huhn [11] gives a far-reaching generalization of this lemma.

By Lemma 8, the equation $\vec{\mathbf{d}}(\vec{u}) = \vec{u}$ says that either $\vec{u}$ is a diamond or it is a singleton. For $\vec{u} = (a, b, c) \in L^3$ and $\vec{v} = (d, e, f) \in L^3$, we define $\vec{\mathbf{m}}(\vec{u}, \vec{v}) = (d^*, e^*, f^*)$:

$$\begin{array}{rclcl} \vec{v}\,' & = & (d', e', f') & := & (c + d, c + e, c + f), \\ \vec{v}\,'' & = & (d'', e'', f'') & := & \vec{\mathbf{d}}(\vec{v}\,'), \\ \vec{v}\,''' & = & (d''', e''', f''') & := & (a + b + e''f'', e'', f''), \\ \vec{\mathbf{m}}(\vec{u}, \vec{v}) = \vec{v}^{\,*} & = & (d^*, e^*, f^*) & := & \vec{\mathbf{d}}(\vec{v}\,'''). \end{array}$$

Our motivation is to reach the northeast situation of Figure 2 or to collapse $\vec{v}$ into a singleton.

Lemma 9. *If at least one of $\vec{v}$, $\vec{v}\,'$, $\vec{v}\,''$, $\vec{v}\,'''$, and $\vec{v}^{\,*}$ has two comparable components, then $\vec{\mathbf{m}}(\vec{u}, \vec{v}) = \vec{v}^{\,*}$ is a singleton.*

Proof. Evident by Lemma 8. □

Now we are in the position to define the lattices L_n, $n \geq 3$. We illustrate this definition with L_3 and L_6 in Figure 1, where the gray squares stand for the diamonds of L_6.

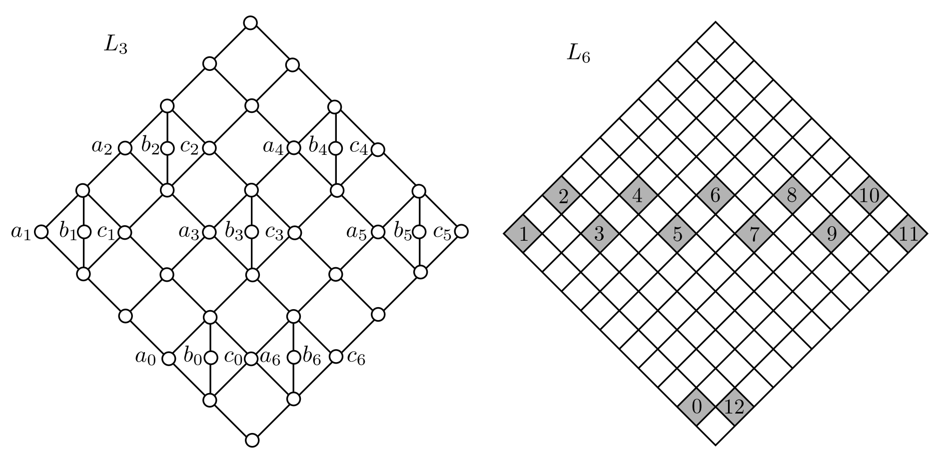

FIGURE 1. L_3 and the scheme of L_6

For each $n \geq 3$, the lattice Frame L_n is the direct square of the $2n$-element chain and L_n has $2n + 1$ diamonds, labeled by $0, 1, \ldots, 2n$. Starting from the 0-th diamond, we make $2n - 3$ northwest steps to reach the first diamond, then two northeast steps to the second one, then two southeast steps to the third one, and so on, finally $2n - 3$ southwest steps to the $2n$-th diamond. Notice that the rightmost element of the 0-th diamond coincides with the leftmost element of the $2n$-th one.

Lemma 10. *Let $n \geq 2$ and $\vec{u} = (a, b, c)$, and let $\vec{v} = (d, e, f)$ belong to the cube of L_n such that $\vec{\mathbf{d}}(\vec{u}) = \vec{u}$. Let $\vec{v}^{\,*} = (d^*, e^*, f^*) := \vec{\mathbf{m}}(\vec{u}, \vec{v})$. Then $\vec{\mathbf{d}}(\vec{v}^{\,*}) = \vec{v}^{\,*}$, and the following three statements hold:*

(a) *If $\vec{u}$ is a diamond whose middle element is c or if $\vec{u}$ is a singleton, then $\vec{v}^{\,*}$ is a singleton. In particular, if $\vec{v}^{\,*}$ is a diamond, then $\vec{u}$ is a diamond as well.*

(b) *If $\vec{v}^*$ is a diamond and c is the rightmost element of $\vec{u}$, then d^* is the leftmost element of $\vec{v}^*$ and the elements a, b, c, d^*, e^*, f^* generate, apart from the a-b and e-f symmetries, either the "northeast" lattice of Figure 2 or its eight-element quotient lattice by the indicated congruence.*

(c) *If $\vec{v}^*$ is a diamond and c is the leftmost element of $\vec{u}$, then d^* is the rightmost element of $\vec{v}^*$ and the elements a, b, c, d^*, e^*, f^* generate, apart from the a-b and e-f symmetries, either the "northwest" lattice of Figure 2 or its eight-element quotient lattice by the indicated congruence.*

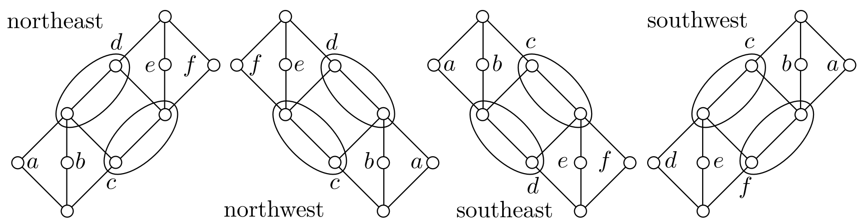

FIGURE 2. Lattices according to compass points

Proof. First, we prove (a). If $\vec{u}$ is a singleton, then $a = b = c \leq d'', e'', f''$. Hence $\vec{v}'''$ has two comparable components and $\vec{v}^*$ is a singleton by Lemma 9. So assume that $\vec{u}$ is a diamond and c is its middle element. If $\vec{v}''$ is a singleton the so is $\vec{v}^*$ by Lemma 9 and there is nothing to prove. Hence we can assume that $\vec{v}''$ is a diamond. Its components form an antichain in the principal filter $\uparrow c$. Since $a + b$ is the only cover of c, this antichain is in $\uparrow(a + b)$. Hence $\vec{v}'''$ has two comparable components, and $\vec{v}^*$ is a singleton by Lemma 9.

Next, we prove (b). Assume that $\vec{v}^*$ is a diamond and c is the rightmost element of $\vec{u}$. In virtue of Lemmas 8 and 9, $\vec{v}''$ is a diamond in $\uparrow c$. Since the M_3-sublattices of L_n are covering sublattices, $a + b$ and $e''f''$ belong to $\operatorname{Frame} L_n$. We have $a + b \not\leq e''f''$, for otherwise $\vec{v}'''$ would have two comparable components, which would contradict Lemma 9. Let h denote the cover of a maximal element in $\uparrow c \setminus \uparrow(a + b)$; c.f. Figure 3, where all but one of the large-sized gray-filled elements are missing. Notice that h is uniquely determined. Since $e''f''$ is in $\uparrow c \setminus \uparrow(a + b)$ and also in

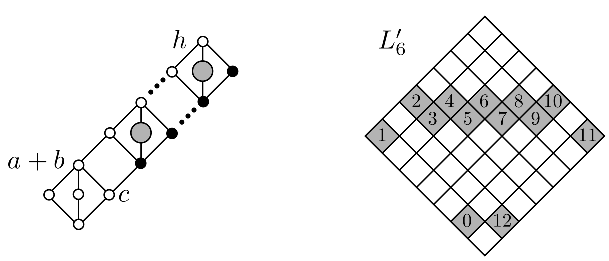

FIGURE 3. A part of L_n and L'_6

$\operatorname{Frame} L_n$, it is one of the black-filled elements in Figure 3. This makes it clear that exactly one component, say x'', of $\vec{u}''$ belongs to the chain $[a + b, h]$. If $x'' \neq d''$, then Lemma 9 leads to a contradiction, for $x''' = x''$ and $d''' = a + b + d''$, belonging to the chain $[a+b,h]$, are comparable. Hence $x''=d''$. It follows easily, either via Figure 3 or using the fact that $e''f''\prec d''$ and $c\prec a+b$, that $\vec{v}^*=\vec{v}''$ with leftmost element $d^*=d''$, $a+b+e^*f^*=d^*$ and $c=(a+b)(e^*f^*)$. This completes the proof of (b).

Finally, (c) follows from (b) via the left-right symmetry. □

We have seen from Lemma 10 that, in some cases, $\vec{\mathbf{m}}(\vec{u},\vec{v})$ produces a diamond which is northeast of $\vec{u}$. Lattice duality and the east-west (that is, right-left) symmetry allow us to navigate in three other directions, northwest, southeast and southwest, too. Therefore we define

$$\begin{aligned}\vec{\mathbf{ne}}((a,b,c),(d,e,f)) &:= \vec{\mathbf{m}}((a,b,c),(d,e,f)),\\ \vec{\mathbf{nw}}((a,b,c),(d,e,f)) &:= \vec{\mathbf{m}}((c,b,a),(f,e,d)),\\ \vec{\mathbf{se}}((a,b,c),(d,e,f)) &:= \vec{\mathbf{m}}^{\bullet}((a,b,c),(d,e,f)),\\ \vec{\mathbf{sw}}((a,b,c),(d,e,f)) &:= \vec{\mathbf{m}}^{\bullet}((c,b,a),(f,e,d)),\end{aligned}$$

where $^{\bullet}$ means that the dual procedure, that is, the lattice terms are dualized.

We consider the following set of variables

$$X=\{x_i : 0\le i\le 2n\}\cup\{y_i : 0\le i\le 2n\}\cup\{z_i : 0\le i\le 2n\}.$$

It will be convenient to gather these variables into vectors

$$\vec{w}_i=(x_i,y_i,z_i)\quad (i=0,1,\dots,2n).$$

We are going to define terms r_i, s_i, t_i over X. These terms will be gathered into vectors

$$\vec{g}_i=(r_i,s_i,t_i)\quad (i=0,1,\dots,2n),$$

and their inductive definition is what follows (compare with Figure 1):

$$\vec{g}_i := \begin{cases}\vec{\mathbf{d}}(\vec{w}_0), & \text{if } i=0;\\ \vec{\mathbf{nw}}(\vec{g}_0,\vec{w}_1), & \text{if } i=1;\\ \vec{\mathbf{ne}}(\vec{g}_{i-1},\vec{w}_i), & \text{if } 2\le i\le 2n-2 \text{ and } i \text{ is even};\\ \vec{\mathbf{se}}(\vec{g}_{i-1},\vec{w}_i), & \text{if } 3\le i\le 2n-1 \text{ and } i \text{ is odd};\\ \vec{\mathbf{sw}}(\vec{g}_{2n-1},\vec{w_{2n}}), & \text{if } i=2n.\end{cases}$$

Finally, let λ_n denote the lattice identity $r_{2n}=s_{2n}$.

4. Proof of Theorem 1 and Corollary 2

Proof of Theorem 1. It is easy to see that, for $i=0,1,\dots,2n$, $r_i(a_i,b_i,c_i)=a_i$, $s_i(a_i,b_i,c_i)=b_i$, and $t_i(a_i,b_i,c_i)=c_i$ hold in L_n. This shows that λ_n does not hold in L_n.

Conversely, assume that $k\ne n$. We have to show that λ_n holds in L_k. Suppose the contrary. Then there are elements of L_k, we denote them by the variables of λ_n, such that evaluating our terms on these elements we have $r_{2n}\ne s_{2n}$. We will make no notational distinction between the terms r_i, s_i, t_i and their values in L_k under this evaluation. Since $\vec{g}_{2n}=(r_{2n},s_{2n},t_{2n})$ is not a singleton, it is a diamond by Lemma 8. Hence a successive application of Lemma 10, combined with duality and symmetry, gives that all the $\vec{g}_i$ are diamonds such that t_i is not the middle element of $\vec{g}_i$, $i=0,1,\dots,2n-1$. Because of the left-right (that is, west-east) symmetry, we can assume that t_0 is the rightmost element of $\vec{g}_0$. Then, using Lemma 10, we obtain that the diamonds $\vec{g}_0, \vec{g}_1, \dots, \vec{g}_{2n}$ are distinct and come after each other in the following directions:

$$\vec{g}_0 \nwarrow \vec{g}_1 \nearrow \vec{g}_2 \searrow \vec{g}_3 \nearrow \vec{g}_4 \searrow \vec{g}_5 \nearrow \vec{g}_6 \searrow \cdots \nearrow \vec{g}_{2n-2} \searrow \vec{g}_{2n-1} \swarrow \vec{g}_{2n}.$$

However, this is impossible in L_k, for $k \neq n$. □

Proof of Corollary 2. Let $P(\mathbf{N}) = \big(P(\mathbf{N}), \subseteq\big)$ be the power set of the set $\mathbf{N} = \{1, 2, \dots\}$. Consider the mapping

$$\varphi\colon P(\mathbf{N}) \to \mathrm{Sub}(\mathbf{Var}(\mathbf{M}_{\mathrm{plan}})), \quad A \mapsto \mathbf{HSP}\{L_{k+2} : k \in A\}.$$

We claim that φ is an order-embedding. It is clear that $A \subseteq B$ implies $\varphi(A) \subseteq \varphi(B)$.

In order to show the reverse implication, assume, by way of contradiction, that $\varphi(A) \subseteq \varphi(B)$, $k \in A$ but $k \notin B$. Then λ_{k+2} holds in all the L_{n+2}, $n \in B$, by Theorem 1. Hence λ_{k+2} holds in $\varphi(B)$. But $L_{k+2} \in \varphi(A) \subseteq \varphi(B)$, so λ_{k+2} holds in L_{k+2}, which contradicts Theorem 1. □

5. Historical comments

Let $\mathbf{M}_{\mathrm{w4}}$ denote the class of modular lattices of width at most four. One of the main results in R. Freese [4] is the complete description of $\mathbf{Si}(\mathbf{Var}(\mathbf{M}_{\mathrm{w4}}))$. G. Grätzer and H. Lakser [8] presents a shorter approach to this description. As an application of the deep structural analysis of R. Freese [4], Theorem 5.9 in [4] asserts more than our Corollary 2: even $\mathbf{Var}(\mathbf{M}_{\mathrm{plan}} \cap \mathbf{M}_{\mathrm{w4}})$ has continuumly many subvarieties. A short new structural approach to Corollary 2 has recently been given by G. Grätzer and R. W. Quackenbush [10].

6. Structural approach versus equational approach

While the previous sections are self-contained, the current one presumes familiarity with some references.

R. Freese [4], G. Grätzer and R. W. Quackenbush [10], and K. A. Baker derive Corollary 2 (or even Theorem 5.9) from *structural* analysis without using or producing identities. However, they also use concrete lattices; see [10] for an account of these lattices, including those of Baker.

Our present argument is *equational*, that is, based on lattice identities. This equational argument could easily be adopted to the previously considered lattices, which are Freese's "snake" lattices and Baker's "snake" lattices. This would lead to an elementary proof of Theorem 5.9 of [4].

In private correspondence, Freese pointed out that his finite snake lattices in [4] and Baker's snake lattices are splitting lattices in $\mathbf{Var}(\mathbf{M}_{\mathrm{plan}})$. This follows from Corollary (3.8) of A. Day [1] and the easy Lemma 5 here. (Even more is true: an appropriate variant of Corollary (5.4) in Day says that these snakes are splitting lattices even in the variety of modular lattices.)

Almost the same idea is valid for our lattices L_n. Let L'_n denote the lattice obtained from L_n by collapsing all pairs of transposed intervals that are intervals of distinct diamonds but not collapsing any diamond. For example, L'_6 is the lattice given in Figure 3 (compare it with Figure 1). Then L'_n is a quotient lattice of L_n and, in the terminology of A. Day [1], L_n is obtained from L'_n via pulling apart all coincident diamond edges. Since there is an evident representation of L_n as a subdirect product of L'_n and two-element lattices, L_n and L'_n generate the same variety. Clearly, L'_n is subdriectly irreducible since it has enough M_3's to guarantee that all prime quotients are projective, therefore L'_n is splitting in $\mathbf{Var}(\mathbf{M}_{\text{plan}})$ by Day's Corollary (3.8) in [1]. (As in the last paragraph of A. Day [1], we can conjecture that L'_n is splitting even in the variety of modular lattices.)

Since Freese's snakes, Baker's snakes, and (supposedly) our L'_n are splitting lattices, one could use the conjugate equations associated with these lattices instead of our λ_n. However, *finding these* conjugate equations seems to be a difficult task. (The difficulty of finding conjugate equations in general is well demonstrated by R. Freese [5] and A. Day and R. Freese [2].) Hence it is natural that λ_n is not the conjugate equation associated with L'_n. Indeed, by rearranging the diamonds in L'_n, it is easy to produce a planar subdirectly irreducible lattice K'_n with $|K'_n| = |L'_n|$ such that λ_n fails in K'_n but (by B. Jónsson [12]) L'_n does not belong to the variety generated by K'_n, because it is not isomorphic with K'_n.

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University of Szeged, Bolyai Institute, Szeged, Aradi vértanúk tere 1, HUNGARY 6720

E-mail address: czedli@math.u-szeged.hu

URL: http://www.math.u-szeged.hu/∼czedli/

University of Szeged, Bolyai Institute, Szeged, Aradi vértanúk tere 1, HUNGARY 6720

E-mail address: mmaroti@math.u-szeged.hu

URL: http://www.math.u-szeged.hu/∼mmaroti/